Def A tracial vNa M is **existentially closed (e.c.)** if: whenever $M \subseteq N$, any quantifier-free formula $\varphi(x,y)$, any $a \in M$, we have

$$\inf_{b \in M} \varphi(a,b)^M = \inf_{c \in N} \varphi(a,c)^N.$$

Lemma M is e.c. iff: whenever $M \subseteq N$, then there is $N \hookrightarrow M^{\mathcal{U}}$ restricting to the diagonal embedding $M \hookrightarrow M^{\mathcal{U}}$.

Facts

- Ec tracial vNas exist.
  If M is a tracial vNa, there is an

e.c. tracial vNa $N \supseteq M$.
If M is sep., can take N sep.

- Ec tracial vNas are II_1 factors. Extra properties: McDuff, all autos approx inner, ...

Def Call a tracial vNa **embeddable** if it embeds in $\mathcal{R}^{\mathcal{U}}$, i.e. if it's a model of $\mathrm{Th}_\forall(\mathcal{R})$.

Prop $\mathcal{R}$ is an e.c embeddable factor.

Pf 1:

$$\begin{tikzcd} \mathcal{R} \arrow[r, phantom, "\subseteq"] \arrow[rr, bend right] & M \arrow[r, leftrightarrow] & \mathcal{R}^{\mathcal{U}} \end{tikzcd}$$

embeddable

may not be the diagonal embedding

Fact: any embedding of $\mathcal{R} \overset{\phi}{\hookrightarrow} \mathcal{R}^{\mathcal{U}}$

is unitarily conjugate to the diagonal embedding, i.e. $\exists u \in \mathcal{U}(\mathcal{R}^{\mathcal{U}})$ s.t. $u\,\phi(x)\,u^* = x \quad \forall x \in \mathcal{R}$. ▨

Digression Kenley Jung: If $N \hookrightarrow \mathcal{R}^{\mathcal{U}}$ is s.t. any two embeddings are unit. conj, then $N \cong \mathcal{R}$.

Atkinson-Elayavalli: If N is emb. and any two embeddings $N \hookrightarrow N^{\mathcal{U}}$ are unit conj, then $N \cong \mathcal{R}$.

AE-G.: If N is emb. and any two embeddings $N \hookrightarrow N^{\mathcal{U}}$ are conj. by an auto, then $N \cong \mathcal{R}$. (Model theory!)

Def A tracial vNa M is **locally universal** if for any N, $N \hookrightarrow M^{\mathcal{U}}$.

CEP: $\mathcal{R}$ is locally universal.

Lemma An e.c. tracial vNa M is locally universal.

Pf: Given any N, want $N \hookrightarrow M^{\mathcal{U}}$.

$$\begin{tikzcd} N \arrow[r, phantom, "\subseteq"] & N \bar{\otimes} M \arrow[r, hook] & M^{\mathcal{U}} \\ & M \arrow[u, phantom, "\cup"] \arrow[ur, hook, "\text{b/c } M \text{ is e.c.}"'] & \end{tikzcd}$$

∎

Cor CEP $\Leftrightarrow$ $\mathcal{R}$ is an e.c. factor.

Pf ($\Leftarrow$) Know $\mathcal{R}$ is an e.c. emb. factor.

CEP $\Rightarrow$ everything is emb.

(⇐) $\mathcal{R}$ e.c. $\Rightarrow$ $\mathcal{R}$ loc univ $\Leftrightarrow$ CEP. $\boxtimes$

Back to games

- 2 players
- Play finite sets of conditions of the form $\varphi(c) < r$, satisfiable
  - c constants from C
  - φ q.f.

- Extend each other's play.
- At the end, built a separable tracial vNa, called the compiled algebra.

Said a property P of tracial vNas is **enforceable** if $\exists$ (player II) has a strategy that forces the compiled algebra to have property P.

Last time: being a II_1 factor is enforceable

Prop Being e.c. is enforceable.

Pf: Enough to show: given any qf $\varphi(x,y)$, constants c, and rational $r>0$, either the compiled structure has no extension with $\inf_y \varphi(c,y)<r$ or else there are c' s.t. $\varphi(c,c')<r$.

Player $\forall$ opens the game with p_0.

If there is a tracial vNa satisfying p_0 and $\inf_y \varphi(c,y)<r$, then there are constants c' s.t. $p_0 \cup \{\varphi(c,c')<r\}$ is a condition and $\exists$ plays it.

Otherwise, every model of p_0 thinks $\inf_y \varphi(c,y) \geq r$ and so the compiled structure thinks that as well. $\square$

Def A separable tracial vNa M is **enforceable** if the property of being $\cong M$ is an enforceable property.

Q Is there an enforceable II_1 factor?

Examples

- The random graph is the enforceable graph.
- The enforceable field of char p is $\mathbb{F}_p$.

Note If T is $\forall\exists$ and has JEP, then M is enforceable iff it is e.c. and embeds in all other e.c. models (e-atomic)

Thm There is no enforceable group.

Pf: If G is the enforceable group, then G embeds in every e.c. group.

By a theorem of Macintyre, every f.g. subgroup of G has solvable word problem.

But every e.c. group has a f.g. subgroup w/ unsolv. word problem. $\not\perp$. $\boxtimes$

Example There is an enf. Banach space, Gurarij Banach space.

Thm $\mathcal{R}$ is the enforceable embeddable factor.

Pf Model theory reason: If P is a $\forall\bigvee\exists$-axiomatizable property ($\forall\vec{x}\bigvee_n \underbrace{\varphi_n(\vec{x})}_{\text{existential}}$) and there is a locally universal object with property P, then P is enforceable.

Us: P = hyperfinite

Player $\forall$ opens with p satisfiable in some emb. M.

Since $M \hookrightarrow \mathcal{R}^{\mathcal{U}}$, p is satisfied in $\mathcal{R}$.

$\exists$ can respond with approximate $p \cup \{(e_{ij})$ are matrix units and c_n is close to some lin comb of e_{ij}'s }

$\therefore$ Being hyperfinite is enf.

But being a II_1 factor is enf.

Use $\mathcal{R}$ is the unique sep. hyperfinite II_1 factor (Murray-von Neumann). $\boxed{\text{//}}$

Cor TFAE:

① CEP

② $\mathcal{R}$ is the enf. II_1 factor

③ Being embeddable is enforceable.

Pf: ① $\Rightarrow$ ② ✓

② $\Rightarrow$ ③ obvious.

③ $\Rightarrow$ ① By ③, there is an e.c factor that is embeddable. This e.c. factor is locally universal, so CEP is true. $\boxed{\text{//}}$

Fact If σ is a sentence, then there is a unique $r_\sigma \in \mathbb{R}$ s.t. $\sigma = r_\sigma$ is enforceable.

If $r_\sigma = \sigma^{\mathcal{R}}$ for every universal σ, then being embeddable would be enforceable.
$\therefore$ $\exists$ universal σ s.t. $r_\sigma \neq \sigma^{\mathcal{R}}$ and $\sigma = r_\sigma$ is enforceable. i.e. o~~ne~~ can enforce the compiled algebra M to satisfy $\sigma^{M} \neq \sigma^{\mathcal{R}}$, so $M \not\hookrightarrow \mathcal{R}^{\mathcal{U}}$.

Open Question Does the enforceable II_1 factor $\mathcal{E}$ $\exists$?

Yes

No

$\neg$CEP.

$\mathcal{E}$ vs. $\mathcal{R}$ Improvement of ...

$\mathcal{E} \hookrightarrow$ every e.c. II_1 factor

$\mathcal{R} \hookrightarrow$ every II_1 factor

$\mathcal{R} \cong \mathcal{R} \bar{\otimes} \mathcal{R}$

$\mathcal{E} \not\cong \mathcal{E} \bar{\otimes} \mathcal{E}$

$\mathcal{E} \not\equiv \mathcal{E} \bar{\otimes} \mathcal{E}$

Atkinson: every emb $\mathcal{R} \hookrightarrow \mathcal{R}^{\mathcal{U}}$ has a lift $\varphi_n : \mathcal{R} \to \mathcal{R}$ that induce $\mathcal{R} \hookrightarrow \mathcal{R}^{\mathcal{U}}$.

Atkinson proved that characterizes $\mathcal{R}$ amongst emb. factors.

G. $\mathcal{E}$ has that property.

$\mathcal{R} \cong L(\Gamma)$ for any ctbl, infinite, ICC <u>amenable</u> group

Is being $\cong$ a group vNa an enforceable property?